ESTIMATES FOR EIGENVALUES OF THE OPERATOR L_r

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Abstract. In this paper, we consider an eigenvalue problem of the elliptic operator
$$L_r = \operatorname{div}(T^r \nabla \cdot)$$
on compact submanifolds in arbitrary codimension of space forms $\mathbb{R}^N(c)$ with $c \geq 0$. Our estimates on eigenvalues are sharp.

1. Introduction

Let $x : M \to \mathbb{R}^N(c)$ be an n-dimensional orientable closed connected submanifold of an N-dimensional space form $\mathbb{R}^N(c)$ of constant sectional curvature c, where $\mathbb{R}^N(c)$ is Euclidean space $\mathbb{R}^N$ when $c = 0$, $\mathbb{R}^N(c)$ is a unit sphere $\mathbb{S}^N$ when $c = 1$, and $\mathbb{R}^N(c)$ is a hyperbolic space $\mathbb{H}^N$ when $c = -1$. Let $\{e_A\}_{A=1}^N$ be an orthonormal basis along M such that $\{e_i\}_{i=1}^n$ are tangent to M and $\{e_\alpha\}_{\alpha=n+1}^N$ are normal to M. Denote by $\{\theta_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=n+1}^N$ the dual frame, respectively. Then we have the following structure equation (see [6]):

$$dx = \sum_i \theta_i e_i, \tag{1.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{\alpha,j} h_{ij}^\alpha \theta_j e_\alpha - c\theta_i x, \tag{1.2}$$

$$de_\alpha = -\sum_{i,j} h_{ij}^\alpha \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta, \tag{1.3}$$

where h_{ij}^α denote the components of the second fundamental form of x. Let $B_{ij} = \sum_{\alpha=n+1}^N h_{ij}^\alpha e_\alpha$. If $r \in \{0, 1, \cdots, n-1\}$ is even, the operator L_r is defined by

$$L_r(f) = \sum_{i,j} T_{ij}^r f_{ij}, \tag{1.4}$$

and

$$L_{r-1}(f) = \sum_{i,j} T_{\alpha ij}^{r-1} f_{ij} e_\alpha. \tag{1.5}$$

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Here T^r is given by

$$T^r_{ij} = \frac{1}{r!} \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_r}} \delta^{j_1 \cdots j_r j}_{i_1 \cdots i_r i} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle; \tag{1.6}$$

$$T^{r-1}_{\alpha ij} = \frac{1}{(r-1)!} \sum_{\substack{i_1 \cdots i_{r-1} \\ j_1 \cdots j_{r-1}}} \delta^{j_1 \cdots j_{r-1} j}_{i_1 \cdots i_{r-1} i} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}} \rangle h^{\alpha}_{i_{r-1} j_{r-1}}, \tag{1.7}$$

$\delta^{j_1 \cdots j_r j}_{i_1 \cdots i_r i}$ is the generalized Kronecker symbols. It has been shown in [6] that T^r is symmetric and divergence-free. When r is even,

$$L_r(f) = \operatorname{div}(T_r \nabla f),$$

and the corresponding rth mean curvature function S_r and $(r+1)$th mean curvature vector field $\mathbf{S}_{r+1}$ are given by

$$\begin{aligned} S_r =& \frac{1}{r!} \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_r}} \delta^{j_1 \cdots j_r}_{i_1 \cdots i_r} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle \\ =& \frac{1}{r} T^{r-1}_{\alpha ij} h^{\alpha}_{ij} \\ =& \binom{n}{r} H_r; \end{aligned} \tag{1.8}$$

$$\begin{aligned} \mathbf{S}_{r+1} =& \frac{1}{(r+1)!} \sum_{\substack{i_1 \cdots i_{r+1} \\ j_1 \cdots j_{r+1}}} \delta^{j_1 \cdots j_{r+1}}_{i_1 \cdots i_{r+1}} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle B_{i_{r+1} j_{r+1}} \\ =& \frac{1}{r+1} T^r_{ij} h^{\alpha}_{ij} e_{\alpha} \\ =& \binom{n}{r+1} \mathbf{H}_{r+1}. \end{aligned} \tag{1.9}$$

It has also been shown in [6] that for any even integer $r \in \{0, 1, \cdots, n-1\}$, we have

$$\operatorname{trace}(T^r) = (n-r) S_r \tag{1.10}$$

and

$$L_r(x) = (r+1)\mathbf{S}_{r+1} - c(n-r) S_r \, x, \tag{1.11}$$

$$L_r(e_\alpha) = -\sum_{i,j,k} T^r_{ij} h^{\alpha}_{ik,j} e_k - \sum_{i,j,k,\beta} T^r_{ij} h^{\alpha}_{ik} h^{\beta}_{jk} e_\beta + c \sum_{i,j} T^r_{ij} h^{\alpha}_{ij} x. \tag{1.12}$$

When M is a hypersurface of a space form, we have

$$L_0(f) = \Delta(f), \qquad L_1(f) = \square(f) = (nH\delta_{ij} - h_{ij}) f_{ij}, \tag{1.13}$$

where the operator $\square$ was introduced by Cheng-Yau in [5] and studied by many mathematicians. In [1], Alencar, do Carmo and Rosenberg generalized Reilly's inequality to more general operators L_r than the Laplacian. That

is, they proved that when M is an orientable closed hypersurface of $\mathbb{R}^{n+1}$ with $H_{r+1} > 0$,

$$\lambda_1^{L_r} \int\limits_M H_r \, dv \le c(r) \int\limits_M H_{r+1}^2 \, dv \tag{1.14}$$

and equality holds precisely if M is a sphere. Here $c(r) = (n-r)\binom{n}{r}$. In [9], Grosjean obtained the following similar optimal upper bound for $\lambda_1^{L_r}$ of closed hypersurfaces of any space form with $H_{r+1} > 0$ and convex isometric immersion x:

$$\lambda_1^{L_r} \operatorname{vol}(M) \le c(r) \int\limits_M \frac{H_{r+1}^2 + cH_r^2}{H_r} \, dv \tag{1.15}$$

and equality holds if and only if $x(M)$ is an umbilical sphere. For eigenvalues of L_r and some important elliptic operators, see also [2–4, 7, 8, 10] and references therein.

In this paper, we assume that L_r is elliptic on M, for some even integer $r \in \{0, 1, \cdots, n-1\}$. The purpose of this paper is to study the following closed eigenvalue problem of the elliptic operator L_r:

$$L_r(u) = -\lambda u \tag{1.16}$$

on compact submanifolds in arbitrary codimension of space forms. We know that the set of eigenvalues consists of a sequence

$$0 = \lambda_0^{L_r} < \lambda_1^{L_r} \le \lambda_2^{L_r} \le \cdots \le \lambda_k^{L_r} \cdots \to +\infty.$$

Denote by u_i the normalized eigenfunction corresponding to $\lambda_i^{L_r}$ such that $\{u_i\}_0^\infty$ becomes an orthonormal basis of $L^2(M)$, that is

$$\begin{cases} L_r(u_i) = -\lambda_i^{L_r} u_i, \\ \int_M u_i u_j \, dv = \delta_{ij}, \quad \text{for any } i, j = 0, 1, \cdots. \end{cases}$$

We will prove the following results:

Theorem 1.1. *Let (M, g) be an n-dimensional orientable closed connected submanifold of a space form $\mathbb{R}^N(c)$ with $c \ge 0$. Assume that L_r is elliptic on M, for some even integer $r \in \{0, 1, \cdots, n-1\}$. Then we have*

$$\lambda_1^{L_r} \int\limits_M H_r \, dv \le c(r) \int\limits_M (|\mathbf{H}_{r+1}|^2 \, dv + c \, H_r^2) \, dv; \tag{1.17}$$

$$\sum_{i=1}^n \sqrt{\lambda_i^{L_r}} \le \frac{n}{\operatorname{vol}(M)} \sqrt{(n-r) \int\limits_M S_r \, dv \int\limits_M (|\mathbf{H}|^2 + c) \, dv}, \tag{1.18}$$

where $c(r) = (n-r)\binom{n}{r}$.

In particular, for $c = 0$, the equality in (1.17) *holds if and only if M is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in* (1.17) *holds if and only if x is r-minimal. For $c = 0$, the equality in* (1.18) *holds if and only if M is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in* (1.18) *holds if and only if x is minimal.*

Using the fact
$$\lambda_1^{L_r} \le \lambda_2^{L_r} \le \cdots \le \lambda_n^{L_r},$$
we have
$$\sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} \ge n\sqrt{\lambda_1^{L_r}}.$$
Therefore, we obtain the following upper bound of the first eigenvalue $\lambda_1^{L_r}$ from (1.18):

Corollary 1.2. *Under the assumption of Theorem 1.1, we have*
$$\lambda_1^{L_r} \le \frac{n-r}{(\mathrm{vol}(M))^2} \int\limits_M S_r\, dv \int\limits_M (|\mathbf{H}|^2 + c)\, dv. \tag{1.19}$$
In particular, for $c = 0$, the equality in (1.19) *holds if and only if M is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in* (1.19) *holds if and only if x is minimal.*

In particular,
$$\sqrt{\lambda_1^{L_r}} \le \sqrt{\lambda_2^{L_r}} \le \cdots \le \sqrt{\lambda_n^{L_r}} < \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}}.$$
Hence, we also obtain

Corollary 1.3. *Under the assumption of Theorem 1.1, we have*
$$\lambda_n^{L_r} < \frac{n^2(n-r)}{(\mathrm{vol}(M))^2} \int\limits_M S_r\, dv \int\limits_M (|\mathbf{H}|^2 + c)\, dv. \tag{1.20}$$

Remark 1.1. When $N = n+1$ and $c = 0$, our estimate (1.17) becomes the result (1.14) of Alencar, do Carmo and Rosenberg in [1]. For $N = n+1$ and $c \ge 0$, our estimate (1.17) seems like the estimate (1.15) of Grosjean in [9]. But our estimate (1.17) is independent of the convex isometric immersion.

Remark 1.2. Clearly, our estimate (1.19) is new. Moreover, we obtain estimates on high order eigenvalues of the elliptic operator L_r on submanifolds of space forms with arbitrary codimension.

2. Proof of results

In order to complete our proof, we need the following lemma:

Lemma 2.1. *Under the assumption of Theorem 1.1, for any function $h_A \in C^2(M)$ satisfying*
$$\int\limits_M h_A u_0 u_B = 0, \quad \text{for } B = 1, \cdots, A-1, \tag{2.1}$$
we have
$$\lambda_A^{L_r} \int\limits_M \langle T^r \nabla h_A, \nabla h_A \rangle\, dv \le \int\limits_M |\mathrm{div}(T^r \nabla h_A)|^2\, dv; \tag{2.2}$$

and

$$\sqrt{\lambda_A^{L_r}}\int\limits_M |\nabla h_A|^2\,dv \le \delta\int\limits_M \langle T^r\nabla h_A, \nabla h_A\rangle\,dv + \frac{1}{4\delta}\int\limits_M (\Delta h_A)^2\,dv, \tag{2.3}$$

where δ is any positive constant.

Proof. We let $\varphi_A = h_A u_0 - u_0 \int\limits_M h_A u_0^2\,dv$. Then

$$\int_M \varphi_A u_0\,dv = 0. \tag{2.4}$$

It has been shown from (2.1) that

$$\int\limits_M \varphi_A u_B\,dv = 0, \quad \text{for } B = 1, \cdots, A-1. \tag{2.5}$$

Hence, we have from the Rayleigh-Ritz inequality

$$\lambda_A^{L_r}\int\limits_M \varphi_A^2\,dv \le -\int\limits_M \varphi_A L_r(\varphi_A)\,dv. \tag{2.6}$$

Since u_0 is a nonzero constant satisfying $u_0^2\,\mathrm{vol}(M) = 1$, and T^r is symmetric and divergence-free, a direct calculation yields

$$\begin{aligned} -\int\limits_M \varphi_A L_r(\varphi_A)\,dv &= -\int\limits_M \varphi_A\,\mathrm{div}(T^r\nabla(h_A u_0))\,dv \\ &= \int\limits_M \langle T^r\nabla(h_A u_0), \nabla(h_A u_0)\rangle\,dv \\ &= u_0^2\int\limits_M \langle T^r\nabla h_A, \nabla h_A\rangle\,dv. \end{aligned} \tag{2.7}$$

Putting (2.7) into the inequality (2.6) gives

$$\lambda_A^{L_r}\int\limits_M \varphi_A^2\,dv \le u_0^2\int\limits_M \langle T^r\nabla h_A, \nabla h_A\rangle\,dv. \tag{2.8}$$

We define

$$\omega_A := -\int\limits_M \varphi_A\,\mathrm{div}(T^r\nabla(h_A u_0))\,dv = u_0^2\int\limits_M \langle T^r\nabla h_A, \nabla h_A\rangle\,dv. \tag{2.9}$$

Then (2.8) gives

$$\lambda_A^{L_r}\int\limits_M \varphi_A^2\,dv \le \omega_A. \tag{2.10}$$

From the Schwarz inequality and (2.10), we obtain

$$\begin{aligned}\lambda_A^{L_r}\omega_A^2 =&\lambda_A^{L_r}\left(\int\limits_M \varphi_A \operatorname{div}(T^r\nabla(h_Au_0))\,dv\right)^2\\ \leq&\lambda_A^{L_r}\left(\int\limits_M \varphi_A^2\,dv\right)\left(\int\limits_M |\operatorname{div}(T^r\nabla(h_Au_0))|^2\,dv\right)\\ \leq&\omega_A\int\limits_M |\operatorname{div}(T^r\nabla(h_Au_0))|^2\,dv,\end{aligned}\tag{2.11}$$

which gives

$$\lambda_A^{L_r}\omega_A \leq \int\limits_M |\operatorname{div}(T^r\nabla(h_Au_0))|^2\,dv.\tag{2.12}$$

Combining (2.9) with (2.12) yields the inequality (2.2).

On the other hand, from the the Stokes formula, one gets

$$\begin{aligned}-u_0\int\limits_M \varphi_A\Delta h_A\,dv =&-\int\limits_M \varphi_A\Delta(h_Au_0)\,dv\\ =&-\int\limits_M \Big(h_Au_0-u_0\int\limits_M h_Au_0^2\,dv\Big)\Delta(h_Au_0)\,dv\\ =&\int\limits_M |\nabla(h_Au_0)|^2\,dv\\ =&u_0^2\int\limits_M |\nabla h_A|^2\,dv.\end{aligned}$$

Therefore, for any positive constant δ, we derive from (2.8)

$$\begin{aligned}\sqrt{\lambda_A^{L_r}}u_0^2\int\limits_M |\nabla h_A|^2\,dv =&-\sqrt{\lambda_A^{L_r}}u_0\int\limits_M \varphi_A\Delta h_A\,dv\\ \leq&\delta\lambda_A^{L_r}\int\limits_M \varphi_A^2\,dv+\frac{1}{4\delta}u_0^2\int\limits_M (\Delta h_A)^2\,dv\\ \leq&\delta u_0^2\int\limits_M \langle T^r\nabla h_A,\nabla h_A\rangle\,dv+\frac{1}{4\delta}u_0^2\int\limits_M (\Delta h_A)^2\,dv.\end{aligned}\tag{2.13}$$

The desired inequality (2.3) is obtained. □

Proof of the estimate (1.17) *in Theorem 1.1.* For $c=0$, according to the orthogonalization of Gram and Schmidt, we get that there exists an orthogonal

matrix $O = (O_A^B)$ such that

$$\sum_{C=1}^{N} \int_M O_A^C x_C u_0 u_B = \sum_{\gamma=1}^{N} O_A^C \int_M x_C u_0 u_B = 0, \quad \text{for } B = 1, \dots, A-1. \tag{2.14}$$

Taking $h_A = \sum_{C=1}^{N} O_A^C x_C$ in (2.2), and summing over A from 1 to N, we obtain

$$\sum_{A=1}^{N} \lambda_A^{L_r} \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv \leq \sum_{A=1}^{N} \int_M |\mathrm{div}(T^r \nabla h_A)|^2 \, dv. \tag{2.15}$$

Since L_r is elliptic, namely T^r is positive definite, we have

$$\sum_{A=1}^{N} \lambda_A^{L_r} \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv \geq \lambda_1^{L_r} \sum_{A=1}^{N} \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv. \tag{2.16}$$

Therefore, from (2.15) and the orthogonal matrix O, we derive

$$\lambda_1^{L_r} \sum_{A=1}^{N} \int_M \langle T^r \nabla x_A, \nabla x_A \rangle \, dv \leq \sum_{A=1}^{N} \int_M |\mathrm{div}(T^r \nabla x_A)|^2 \, dv. \tag{2.17}$$

Let $E_1, \cdots, E_N$ be a canonical orthonormal basis of $\mathbb{R}^N$, then $x_A = \langle E_A, x \rangle$ and

$$\nabla(x_A) = \langle E_A, e_i \rangle e_i = E_A^{\top}, \tag{2.18}$$

where $\top$ denote the tangent projection to M. Therefore,

$$\begin{aligned}
\sum_{A=1}^{N} \langle T^r \nabla x_A, \nabla x_A \rangle &= \sum_{A=1}^{N} T_{ij}^r \langle E_A, e_i \rangle \langle E_A, e_j \rangle \\
&= T_{ij}^r \langle e_i, e_j \rangle \\
&= \mathrm{trace}(T^r) \\
&= (n-r) S_r,
\end{aligned}$$

which shows that $S_r > 0$ since T^r is positive definite. Using the definition of L_r, we have

$$\sum_{A=1}^{N} |\mathrm{div}(T^r \nabla x_A)|^2 = \sum_{A=1}^{N} \langle E_A, L_r(x) \rangle^2 = |L_r(x)|^2 = (r+1)^2 |\mathbf{S}_{r+1}|^2.$$

Thus, we derive from (2.17)

$$(n-r)\lambda_1^{L_r} \int_M S_r \, dv \leq (r+1)^2 \int_M |\mathbf{S}_{r+1}|^2 \, dv. \tag{2.19}$$

By virtue of the relationships between S_r and H_r, we deduce to

$$\lambda_1^{L_r} \int_M H_r \, dv \leq c(r) \int_M |\mathbf{H}_{r+1}|^2 \, dv. \tag{2.20}$$

When $c=1$,

$$\mathbb{S}^N=\{x\in\mathbb{R}^{N+1};\ |x|^2=x_0^2+x_1^2+\cdots x_N^2=\frac{1}{c}\}.$$

Using the similar method, we can derive

$$\lambda_1^{L_r}\sum_{A=0}^{N}\int_M\langle T^r\nabla x_A,\nabla x_A\rangle\,dv\le\sum_{A=0}^{N}\int_M|\mathrm{div}(T^r\nabla x_A)|^2\,dv. \tag{2.21}$$

Putting

$$\sum_{A=0}^{N}\langle T^r\nabla x_A,\nabla x_A\rangle=\mathrm{trace}(T^r)=(n-r)S_r$$

and

$$\begin{aligned}\sum_{A=0}^{N}|\mathrm{div}(T^r\nabla x_A)|^2=&|L_r(x)|^2\\ =&(r+1)^2|\mathbf{S}_{r+1}|^2+c^2(n-r)^2S_r^2|x|^2\\ =&(r+1)^2|\mathbf{S}_{r+1}|^2+c(n-r)^2S_r^2\end{aligned}$$

into (2.21) gives

$$(n-r)\lambda_1^{L_r}\int_M S_r\,dv\le\int_M[(r+1)^2|\mathbf{S}_{r+1}|^2+c(n-r)^2S_r^2]\,dv. \tag{2.22}$$

Hence, the desired estimate (1.17) is derived.

Next, we consider the case that equalities occur. If $c\ge0$ and the equality in (1.17) holds, then inequalities (2.6), (2.11) and (2.16) become equalities. Hence, we have

$$\lambda_1^{L_r}=\lambda_2^{L_r}=\cdots=\lambda_N^{L_r}=\mu; \tag{2.23}$$

$$L_r(\varphi_A)=-\mu\,\varphi_A, \tag{2.24}$$

where μ is a constant. When $c=0$, from (2.24) and (1.11), we can infer that the vector field $\varphi=(\varphi_1,\cdots,\varphi_N)$ is parallel with $\mathbf{S}_{r+1}$. Thus, we obtain

$$\frac{1}{2}(|\varphi|^2)_{,i}=\langle e_i,\varphi\rangle=0, \tag{2.25}$$

which shows that $|\varphi|^2$ is constant. Hence M is a sphere in $\mathbb{R}^{n+1}$. When $c=1$ and the equality in (1.17) holds, it is easy to see that $\mathbf{S}_{r+1}=0$ by combining (2.24) with (1.11). That is to say that x is r-minimal.

□

Proof of the estimate (1.18) *in Theorem 1.1.* For $c=0$, we taking $h_A=\sum_{C=1}^N O_A^C x_C$ in (2.3), where the matric O is given by (2.14). From (2.18), we get

$$|\nabla h_A|^2=|\nabla x_A|^2=|E_A^\top|^2\le|E_A|^2=1,\quad\forall\ A, \tag{2.26}$$

and

$$\sum_{A=1}^{N} |\nabla h_A|^2 = \sum_{A=1}^{N} |\nabla x_A|^2 = n. \tag{2.27}$$

Thus, we infer

$$\begin{aligned}
&\sum_{A=1}^{N} \sqrt{\lambda_A^{L_r}} |\nabla h_A|^2 \\
&\geq \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^{L_r}} \sum_{\alpha=n+1}^{N} |\nabla h_\alpha|^2 \\
&= \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^{L_r}} \left(n - \sum_{j=1}^{n} |\nabla h_j|^2 \right) \\
&= \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^{L_r}} \sum_{j=1}^{n} (1 - |\nabla h_j|^2) \\
&\geq \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} |\nabla h_i|^2 + \sum_{j=1}^{n} \sqrt{\lambda_j^{L_r}} (1 - |\nabla h_j|^2) \\
&= \sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}}.
\end{aligned} \tag{2.28}$$

From (1.11), we have $\sum_{A=1}^{N} (\Delta h_A)^2 = |\Delta(x)|^2 = |\mathbf{S}_1|^2 = n^2 |\mathbf{H}|^2$. Hence, taking sum on A from 1 to N for (2.3), we have

$$\sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} \operatorname{vol}(M) \leq \delta(n-r) \int_M S_r \, dv + \frac{n^2}{4\delta} \int_M |\mathbf{H}|^2 \, dv. \tag{2.29}$$

Minimizing the right hand side of (2.29) by taking

$$\delta = \frac{n}{2} \sqrt{\frac{\int_M |\mathbf{H}|^2 \, dv}{(n-r) \int_M S_r \, dv}}$$

yields

$$\sum_{i=1}^{n} \sqrt{\lambda_i^{L_r}} \operatorname{vol}(M) \leq n \sqrt{(n-r) \int_M S_r \, dv \int_M |\mathbf{H}|^2 \, dv}. \tag{2.30}$$

When $c = 1$, the proof is similar. We omit it here.

When $c \geq 0$ and the equality in (1.18) holds, then inequalities (2.6), (2.13) and (2.28) become equalities. Hence, we have

$$\lambda_1^{L_r} = \lambda_2^{L_r} = \cdots = \lambda_N^{L_r} = \mu; \tag{2.31}$$

$$\Delta(\varphi_A) = -\mu \varphi_A. \tag{2.32}$$

Similarly, we infer that, for $c = 0$, the equality in (1.18) holds if and only if M is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in (1.18) holds if and only if x is minimal.

$\square$

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